Assignment Nr. 1

due 25 October

Problem 1

Expand the following equations for an index range of three, i.e. $i, j = 1, 2, 3$:

(a) $A_{ij}x_j + b_i = 0$,

(b) $\Phi = C_{ij}x_ix_j$,

(c) $\Psi = T_{ii}S_{jj}$.

Problem 2

Verify the following identities:

(a) $\delta_{ii} = 3$,

(b) $A_{ij}\delta_{ij} = A_{ii}$,

(c) $\delta_{ij}\varepsilon_{ijk} = 0$,

(d) $\varepsilon_{ijk}\varepsilon_{ijk} = 6$,

(e) $\varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{km}$,

(f) $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ ('ε-δ identity') .

Problem 3

(a) Expand and simplify the expression $A_{ij}x_ix_j$, where $i, j = 1, 2, 3$ and

  (i) A_{ij} is symmetric,

  (ii) A_{ij} is skew-symmetric.

(b) Let A_{ij} be symmetric and B_{ij} skew-symmetric. Show that $A_{ij}B_{ij} = 0$.

Problem 4

Recall that the vector product of two vectors $\boldsymbol{u} = u_i \boldsymbol{e}_i$ and $\boldsymbol{v} = v_i \boldsymbol{e}_i$ is a quantity $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$ with components

$$w_1 = u_2 v_3 - u_3 v_2 \ , \quad w_2 = u_3 v_1 - u_1 v_3 \ , \quad w_3 = u_1 v_2 - u_2 v_1 \ ,$$

with reference to the right-hand orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$.

(a) Verify that $w_i = \varepsilon_{ijk} u_j v_k$.

(b) Show that for any three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{u})\boldsymbol{w} \quad .$$

(hint: use the ε-δ identity.)

Problem 5

Use indical notation to verify the following identities:

(a) $\nabla(\phi \boldsymbol{v}) = \phi \nabla \boldsymbol{v} + \nabla \phi \otimes \boldsymbol{v}$,

(b) $\nabla(\boldsymbol{v} \cdot \boldsymbol{v}) = 2\boldsymbol{v} \cdot (\nabla \boldsymbol{v})^T = 2(\nabla \boldsymbol{v}) \cdot \boldsymbol{v}$,

(c) $\nabla \times \nabla \phi = \boldsymbol{0}$,

(d) $\nabla \cdot \nabla \times \boldsymbol{v} = 0$,

(e) $\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla \cdot (\nabla \boldsymbol{v})$,

where ϕ denotes a scalar field and $\boldsymbol{v}$ a vector field in E^3. (hint: for the last relation, use the ε-δ identity.) For each case, indicate if the result is a scalar, a vector, or a second-order tensor.

Problem 6

For arbitrary ω_k, let the components of a tensor $\boldsymbol{W}$ be given by

$$W_{ij} = -\frac{1}{2} \varepsilon_{ijk} \omega_k \quad . \tag{1}$$

(a) Show that $\boldsymbol{W}$ is a skew-symmetric tensor.

(b) Using the relation $\varepsilon_{ijk} \varepsilon_{ijl} = 2\delta_{kl}$ (see above), show that (1) can be inverted to yield

$$\omega_i = \varepsilon_{ijk} W_{kj} \quad . \tag{2}$$