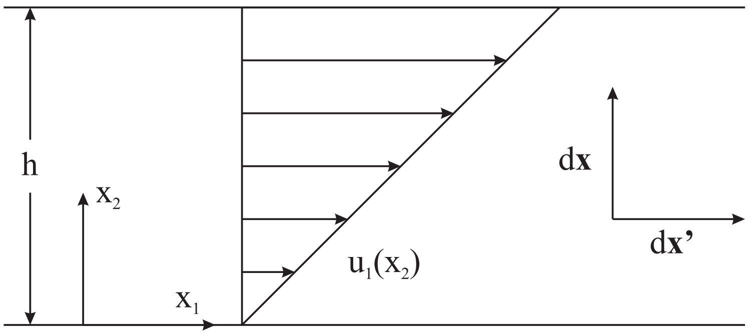
Assignment Nr. 3

due November 22th

Problem 1



Consider a shear flow between two tangentially moving plates (Couette flow) with velocity given by

$$u_1 = \frac{U}{h} x_2 \ , \quad u_2 = u_3 = 0 \ ,$$

where U is the constant velocity difference between the moving plates, and h their constant distance.

(a) Compute the components of the velocity gradient, $\partial u_i / \partial x_j$, in Cartesian coordinates. Use the result to compute the symmetric and anti-symmetric parts, S_{ij} and W_{ij}, of the velocity gradient. Here and in the follwing, it is sufficient to write down all expressions in two dimensions (x_1, x_2) only.

(b) Compute the rate of strain of the two line elements, $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\boldsymbol{x}'$, oriented in the x_2 and x_1 directions, respectively.

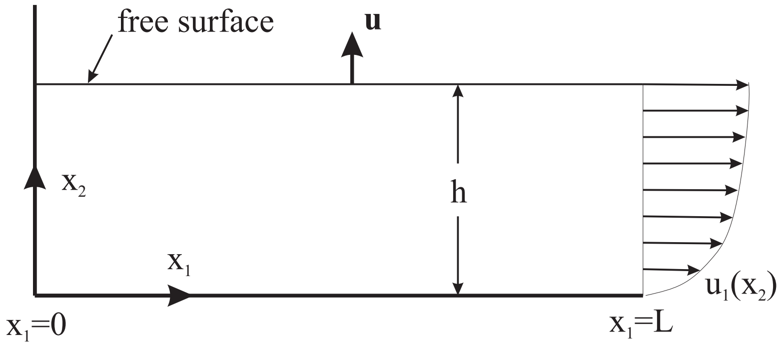
(c) Compute the rate at which the angle between the two line elements changes (shear rate).

(d) Compute the average angular velocity of the two line elements (rotation rate).

(e) Compute the eigenvalues and eigenvectors of the rate of deformation tensor, S_{ij}. What is the angle between these eigenvectors and the x-axis? (Hint: it is sufficient to consider only the 2 by 2 matrix of the plane problem in the x_1 and x_2 directions. Since S_{ij} is real and symmetric, the two eigenvectors are orthogonal.)

(f) What is the significance of the two directions given by the two eigenvectors? How do you think two infinitesimal line elements pointing in these directions are deformed?

Problem 2



The figure shows an estuary (like the Elbe river mouth) with idealized geometry. It consists of a channel of length L that is closed by a vertical wall at $x_1 = 0$ and has an open boundary with the ocean at $x_1 = L$. The flat bottom of the estuary is located at $x_2 = 0$ and the free surface at $x_2 = h$. Due to tidal motions, the position of the horizontal free surface, $h(t)$, is a function of time, t. It is assumed to be described by the formula

$$h(t) = h_0 + a \sin \frac{2\pi t}{T} , \tag{1}$$

where h_0 is the mean water level, a the amplitude of the tide, and T the tidal period. Assume that at $x_1 = L$, the profile of the horizontal velocity is given by

$$u_1(x_2) = U(t) \ln \left(\frac{x_2 + z_0}{z_0} \right) , \tag{2}$$

with $U(t)$ defining an unknown function of time. This is a typical profile for a turbulent channel flow, where the so-called "bottom roughness", z_0, is assumed to be known.

(a) Show that for homogeneous, constant density, $\rho = \rho_0$, the integral form of the balance of mass can be written as

$$\int_A \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}A = 0 , \tag{3}$$

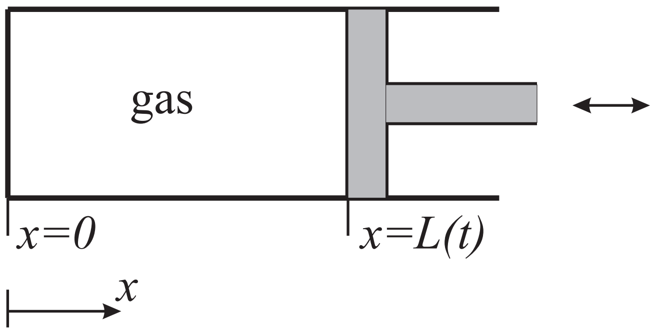
where the surface, A, consists of the bottom, the left closed boundary, the right open boundary, and the free surface.

(b) Evaluate (3) formally, taking into account all four surfaces.

(c) From the solution of (3), derive an expression for the function $U(t)$ in (2).

(d) Use this result to plot profiles of the velocity $u_1(x_2)$ at $x_1 = L$ for $t = T/4$, $T/2$, $3T/4$, and T, using the following numerical values: $L = 100$ km, $h_0 = 50$ m, $a = 5$ m, $z_0 = 0.01$ m, and $T = 12.42$ h. This corresponds to a typical semi-diurnal tide in some estuaries in the North Sea.

Problem 3



Consider a cylindrical tube with constant cross-sectional area, A, that is filled with a compressible gas. The tube is closed at the left end, $x = 0$, such that the gas can be compressed by a moving piston located at $x = L(t)$. The position of the piston is a function of time, t, and assumed to be given by

$$L(t) = \frac{L_0}{2 - \cos \omega t} \;, \tag{4}$$

where the constant angular velocity, ω, and the constant reference position, L_0, are known.

(a) Assuming that the density ρ only depends on t and x, show that the integral balance of mass for the material volume of gas inside the tube is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{L(t)} \rho \,\mathrm{d}x = 0 \quad . \tag{5}$$

(b) Show that *one* possible solution of (5) corresponds to

$$\rho = \rho_0 (2 - \cos \omega t) \;, \tag{6}$$

where ρ_0 is an arbitrary, but constant reference density (hint: use Leibniz' integration rule). This solution is valid only if the speed of the piston, $\dot{L}$, is small compared to the speed of sound. In this case, no sound waves are generated by the piston, and ρ can be assumed to be homogeneous.

(c) Show that for one-dimensional problems, like the one at hand, the general form of the local balance of mass adopts the simple form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \quad . \tag{7}$$

(d) Insert (6) into (7) to obtain an expression for the divergence of the velocity, which is $\partial u / \partial x$ in one-dimensional flows. Integrate the expression for $\partial u / \partial x$ to obtain the spatial distribution of the velocity, $u(x, t)$, inside the tube. Note that at the closed end $u(0, t) = 0$.

(e) Plot L, ρ, and the divergence of the velocity, $\partial u / \partial x$, for $0 \leq t \leq 2\pi$ for $L_0 = \rho_0 = 1$, and $\omega = 2$. Mark periods of positive divergence and periods of decreasing density. Do these periods coincide? Should they?