Assignment Nr. 4

due 08 June

Problem 1

Download the time series of the three velocity components u_1, u_2, and u_3 measured with an accoustic current meter in a turbulent bottom boundary current in the Western Baltic Sea. In this problem, $u = u_1$ and $v = u_2$ denote the horizontal and $w = u_3$ the vertical current components with units [m/s], sampled at a frequency of $f = 25$ Hz.

(a) Compute the temporal mean values, $\overline{u}_i$, of all components.

(b) Compute the angle between the horizontal mean current vector and the x-axis. Introduce a new, horizontally rotated coordinate system such that the x-axis points into the direction of the mean current vector. Recompute the horizontal velocity components with respect to this new coordinate system. So from now on, $u = u_1$ is the component pointing into the direction of the mean flow, and $v = u_2$ the component normal to it. The vertical velocity component $w = u_3$ is not affected by this horizontal rotation.

(c) Re-compute $\overline{u}_i$ with respect to the rotated coordinate system, and verify that $\overline{v} = \overline{u}_2 = 0$. Provide three individiual plots of the u_i as functions of time, and add lines indicating the mean values, respectively.

(d) Compute the fluctuating parts of the velocity components, $u'_i = u_i - \overline{u}_i$, as well as their variances and standard deviations. Provide three individual plots of u'_i as functions of time, and add lines indicating $\pm$ one standard deviation, respectively.

(e) Based on temporal averaging, compute the components of the Reynolds-stress tensor, $\overline{u'_i u'_j}$, from the variances and co-variances of the velocity fluctuations.

(f) Reject or confirm the following statements (justify your answers):

- Turbulence at the observation site is isotropic.
- Compared to isotropic turbulence, the streamwise velocity variance, $\overline{u_1'^2}$, is increased in the given boundary layer.
- Compared to isotropic turbulence, the vertical velocity variance, $\overline{u_3'^2}$, is increased in the given boundary layer.
- The horizontal component of the Reynolds stress is aligned with the horizontal mean velocity vector to good accuracy. (Hint: The components of the horizontal Reynolds stress vector are: $\overline{u'_1 u'_3}$ and $\overline{u'_2 u'_3}$. Compare the direction of this vector with the horizontal mean velocity.)

Problem 2

Show, term by term, how an equation for the turbulent kinetic energy (TKE) of the following form can be derived:

$$\begin{aligned}\frac{\partial k}{\partial t} &+ \frac{\partial}{\partial x_j}\left(\bar{u}_j k + \frac{1}{2}\langle u_i' u_i' u_j' \rangle - 2\nu \langle u_i' S_{ij}' \rangle + \frac{\langle u_j' p' \rangle}{\rho_0}\right) \\ &= -\langle u_i' u_j' \rangle \bar{S}_{ij} - \frac{g}{\rho_0}\langle u_3' \rho' \rangle - 2\nu \langle S_{ij}' S_{ij}' \rangle \ ,\end{aligned} \tag{1}$$

where the notation from the lecture notes has been used (hint: start by computing the dot product of $\boldsymbol{u'}$ and Eq. (2.1) in the lecture notes). Which of the following statements are true (explain your answers):

(a) k is always positive.

(b) The shear production term $-\langle u_i' u_j' \rangle \bar{S}_{ij}$ is always positive (thus leading to increasing k).

(c) In homogeneous, unstratified turbulence without mean shear, k always increases.

(d) The TKE in isotropic turbulence always decays.

Problem 3

Now let's derive a simplified version of (1) by assuming that turbulence is statistically stationary and that the transport terms can be ignored. Let's further assume that the turbulent buoyancy flux is strictly vertical, and that the mean velocity vector points into the horizontal direction $x = x_1$ everywhere and only varies in the vertical direction $z = x_3$.

(a) Show that under the above conditions, the transport equation for the turbulent kinetic energy in (1) reduces to

$$P + G = \varepsilon \ , \tag{2}$$

where P denotes the *shear production*,

$$P = -\langle u' w' \rangle \frac{\partial \langle u \rangle}{\partial z} \ , \tag{3}$$

G the turbulent *buoyancy flux* (or *buoyancy production*),

$$G = -\frac{g}{\rho_0}\langle w' \rho' \rangle \ , \tag{4}$$

and the dissipation is defined as

$$\varepsilon = 2\nu \langle S_{ij}' S_{ij}' \rangle \quad . \tag{5}$$

The special *equilibrium form* of the TKE budget in (2) is a useful and frequently-used first-order approximation of the full TKE budget in many oceanographic and atmospheric turbulent boundary-layer flows.

(b) Show that (2) can also be written as

$$G = -\frac{R_f}{1 - R_f}\varepsilon = -\gamma\varepsilon \ , \tag{6}$$

where $R_f = -G/P$ is the *Flux Richardson number* and $\gamma = R_f/(1 - R_f)$ the *flux coefficient*. Recall that in most stably stratified flows we have $G < 0$ (downward buoyancy flux) and thus $\gamma > 0$ and $R_f > 0$ (for $R_f < 1$).

Here is what you learn from this excercise. Many experiments have shown that in stably stratified turbulence, the flux Richardson number is approximately constant and around $R_f = 0.17$ such that $\gamma = 0.2$. The (downward) turbulent buoyancy flux is therefore about 20% of the dissipation rate ε. As ε is much easier to measure than G, this simple relation is often used to estimate G. In oceanography, this approach is called the "Osborn model". The flux Richardson number R_f is sometimes considered as a *mixing efficiency* because it compares the work that turbulence performs against gravity (i.e., the turbulent buoyancy flux) with the work required to sustain turbulence. The larger the buoyancy flux for a given value of P, the more "efficient" the mixing process.